Stiffness Matrix Adjustment Using Mode Data

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A procedure is introduced that uses, in addition to mode data, structural connectivity information to optimally adjust deficient stiffness matrices. The adjustments performed are such that the percentage change to each stiffness coefficient is minimized. The physical configuration of the analytical model is preserved and the adjusted model will exactly reproduce the modes used in the identification. The theoretical development is presented and the procedure is demonstrated by numerical simulation of a test problem.

Nomenclature

$[A]$ = see Eq. (14)
$\{A\}$ = vector of $[A]$ elements, see Eq. (16)
$\{D^j\}_i$ = ith column of $[\phi]^T[\hat{\Phi}^j]$
E = $[M][\phi][\omega_n^2]$
$[G^i]$ = submatrix i of $[\alpha]$
$[H]_{ij}$ = submatrix ij of $[\beta]$
$[I]$ = identity matrix
$[\hat{I}]$ = see Eq. (4)
$[k]$ = analytical stiffness matrix
k_{ij} = element ij of $[k]$
$[K]$ = adjusted analytical stiffness matrix
K_{ij} = element ij of $[K]$
L = Lagrange function
$[M]$ = mass matrix
m_i = diagonal element ii of $[M]$
$[0]$ = null matrix
$[\alpha]$ = defined by Eq. (18)
$[\beta]$ = defined by Eq. (19)
$[\gamma]$ = matrix of stiffness matrix adjustment coefficients
γ_{ij} = element ij of $[\gamma]$
ϵ = error function
λ_{ij} = Lagrange multiplier
$[\lambda]$ = matrix of Lagrange multipliers λ_{ij}
$\{\lambda\}$ = vector of $[\lambda]$ elements, see Eq. (17)
$[\mu]$ = matrix of Lagrange multipliers μ_{ij}
μ_{ij} = Lagrange multiplier
$\{\rho\}$ = transformed $\{\bar{\lambda}\}$, see Eq. (20)
$[\phi]$ = matrix of normal mode vectors
$\{\phi\}_i$ = ith normal mode vector
$\{\phi\}_j$ = jth column of $[\phi]^T$
ϕ_{ij} = element ij of $[\phi]$
$[\phi^m]$ = matrix of measured mode vectors
$[\phi^c]$ = matrix of analytically orthogonalized measured mode vectors
$[\Phi]$ = $[k]\odot[k]$
$[\hat{\Phi}^i]$ = diagonal matrix whose diagonal elements are the ith row of $[\Phi]$
$[\psi]$ = eigenvectors of $[\alpha]+[\beta]$ associated with $[\Omega]$
$[\omega_n^2]$ = diagonal matrix of circular frequencies squared
$[\Omega]$ = nonzero eigenvalues of $[\alpha]+[\beta]$
$\odot$ = element-by-element matrix multiplication operator

Introduction

ACCURATE dynamic models are required to establish the dynamic response of complex satellite structures. Unfortunately, analytical dynamic models of complex structures agree closely with properly measured mode data only in the first few modes. The effect of this deficiency can be minimized by directly using the measured modes as the dynamic model. This approach has been used successfully on numerous satellite programs. Another approach, generally referred to as system identification, is to adjust the analytical dynamic model in an attempt to improve the correlation between the analytical and empirical modes. A relatively large quantity of work has been published in this field. However, no procedure developed to date has gained wide acceptance in the community.

Numerous goals and approaches to the identification problem are presented in the literature. For example, Rodden[1] published a procedure for establishing structural influence coefficients from modes of an effectively unconstrained structure. Subsequently, using optimization theory, Hall[2] formulated a procedure that established a stiffness matrix such that the resulting analytical modes matched selected empirical modes in a least-squares sense. The mass matrix was assumed to be exact. A procedure to adjust an analytical mass matrix and to derive an approximate stiffness matrix was published by Berman and Flannelly.[3] Ross[4] introduced a procedure for deriving both the mass and stiffness matrices from measured natural frequencies and a square modal matrix composed of measured mode vectors supplemented by arbitrary linearly independent vectors. A similar concept was recently published by Zak,[5] who supplements the measured mode data with information from analytically predicted modes. Iterative procedures that employ statistical parameter estimation to adjust analytical models have been published by Collins et al.[6] and Lee and Hasselman.[7] In addition, identification procedures based on matrix perturbation theory have been proposed by Chen and Wada,[8] Chen and Garba,[9] and Chen et al.[10]

Identification procedures have also been developed using constrained minimization theory. Baruch and Bar Itzhack introduced formulations to adjust analytical stiffness[11] and flexibility[12] matrices such that the resulting analytical dynamic model modes are identical to the analytically orthogonalized test modes used in the identification. These procedures assume that the mass matrix is correct. Berman[13] introduced a formulation that modifies the mass matrix and assumes that the measured modes are exact. Subsequently, Berman and Nagy[14] combined the mass matrix adjustment procedure of Ref. 13 with the stiffness matrix adjustment procedure of Ref. 11 to establish the so-called analytical model improvement (AMI) procedure. This procedure will yield a model whose modes agree exactly with those used in the identification. However, as demonstrated in Ref. 10, the analytical mass and stiffness matrices can be dramatically altered. Particularly troublesome is the modification of stiffness coefficients from values of zero to large magnitude nonzero values. Clearly, the introduction of load paths that do not exist in the actual hardware is undesirable.

It is the purpose of this discussion to introduce a procedure that uses, in addition to mode data, structural connectivity

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information to optimally adjust deficient stiffness matrices. Analytical dynamic models of complex structures typically involve several hundred degrees of freedom. However, at best, only a few dozen measured modes will be available. Thus, the number of stiffness coefficients will greatly exceed the number of equations provided by the measured mode data. It is reasonable to expect that a more accurate identification will result if the ratio of stiffness coefficients to available equations is reduced. This can be accomplished by supplementing the measured mode data with structural connectivity information. Of course, by doing so, we assume that the connectivity of the analytical model is correct.

The stiffness, K, matrix adjustment (KMA) procedure will be developed using constrained minimization theory. An important part of the development is the selection of the error function to be minimized. In Refs. 11 and 14, it is suggested that the appropriate error function should involve the mass-weighted difference between the analytical and adjusted stiffness matrix coefficients. However, in complex spacecraft structures, for example, nonstructural items such as electronic boxes, batteries, and propellant generally account for over 80% of the total mass. Therefore, it is inappropriate to weigh the stiffness coefficient changes by any function involving the mass matrix. In addition, the error function, as defined in Refs. 11 and 14, can result in substantially greater percentage changes occurring in stiffness coefficients with numerically small values than in coefficients with numerically large values. This bias is difficult to justify.

Based on the preceding discussion, it must be concluded that the error function proposed in Refs. 11 and 14 is not suitable for our purposes and, therefore, a new error function must be established. It is proposed that the appropriate error function be independent of the system mass properties and stiffness coefficient magnitudes. Furthermore, minimization of the error function should minimize the percentage change to each stiffness coefficient. The optimally adjusted stiffness matrix can then be obtained by minimizing this error function subject to symmetry constraints, connectivity constraints, and constraints derived from a system of force equilibrium equations.

Theoretical Development

The preceding discussion indicated that it is reasonable to expect that a more accurate stiffness matrix adjustment will result if, in addition to measured mode data, structural connectivity information is used. This can be accomplished by insisting that all coefficients with values of zero in the original stiffness matrix also have values of zero in the adjusted stiffness matrix. Mathematically, this can be achieved by insisting that the adjusted stiffness matrix $[K]$ be related to the original stiffness matrix $[k]$ by

$$[K]=[k]\odot[\gamma] \tag{1}$$

where $[\gamma]$ is a matrix to be determined and the operator $\odot$ defines the following element operations:

$$K_{ij}=k_{ij}\gamma_{ij} \tag{2}$$

Therefore, if k_{ij} has a value of zero, K_{ij} will also have a value of zero. Since extensive use will be made of the element-by-element (scaler) matrix multiplication operator $\odot$, its relevant properties are presented in the Appendix.

To minimize unrealistic changes in stiffness coefficients, the error function to be minimized must be independent of the stiffness coefficient magnitudes.[15] This can be accomplished by defining the error function as

$$\epsilon=\|[\hat{I}]-[\hat{I}]\odot[\gamma]\| = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\hat{I}_{ij}-\hat{I}_{ij}\gamma_{ij}\right)^2 \tag{3}$$

where

$$\hat{I}_{ij}=1 \quad \text{if } k_{ij}\neq 0$$
$$=0 \quad \text{if } k_{ij}=0 \tag{4}$$

The adjusted stiffness matrix can now be obtained by establishing the $[\gamma]$ that minimizes ϵ and satisfies the following constraints

$$-[M][\phi][\omega_n^2]+([k]\odot[\gamma])[\phi]=[0] \tag{5}$$

$$[\gamma]-[\gamma]^T=[0] \tag{6}$$

where $[M]$ is the system mass matrix (n,n) [( ) designates dimension of the matrix], $[\phi]$ the system mode shapes (n,m), and $[\omega_n^2]$ the diagonal matrix of circular frequencies squared (m,m).

Equation (5) introduces a set of force equilibrium constraints. If the mode shapes in Eq. (5) exhibit the orthogonality associated with normal modes, i.e.,

$$[\phi]^T[M][\phi]=[I] \tag{7}$$

and the total number of constraints does not exceed the number of nonzero stiffness coefficients, the adjusted model will exactly reproduce the modes used in the identification. However, because of limitations in experimental apparatus and inaccuracies in the analytical mass matrix, Eq. (7), in practice, will not be satisfied exactly. Therefore, it will be necessary to analytically adjust either the empirical modes,[11,17] the mass matrix,[13] or both, such that Eq. (7) is satisfied. This point is discussed in more detail in the next section. It should be noted, however, that the formulation of the KMA procedure does not require that Eq. (7) be satisfied. Indeed, if Eq. (7) is not satisfied, the adjusted model will still nearly satisfy Eq. (5), but will not necessarily reproduce the modes used in the identification.

The method of Lagrange multipliers[16] will be used to incorporate in the minimization of ϵ the constraints defined by Eqs. (5) and (6). We begin by forming the Lagrange function L

$$L=\epsilon+\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda_{ij}\left(\sum_{l=1}^{n}k_{il}\gamma_{il}\phi_{lj}\right)-E+\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_{ij}(\gamma_{ij}-\gamma_{ji}) \tag{8}$$

where λ_{ij} and μ_{ij} are the Lagrange multipliers. The quantity E represents the first term of Eq. (5), which is not a function of γ_{ij} and therefore does not need to be explicitly defined. Proceeding, we take the partial derivative of L with respect to each γ_{ij}. These derivatives are set equal to zero to obtain a set of $n\times n$ equations that the γ_{ij} must satisfy for L to be a minimum. Expressing these equations in matrix notation, we obtain

$$-2([\hat{I}]-[\gamma])+[k]\odot\left([\lambda][\phi]^T\right)+[\mu]=[0] \tag{9}$$

Equations (5), (6), and (9) represent the complete system of equations needed to determine the unknowns $[\lambda]$, $[\mu]$, and $[\gamma]$. Since $[\mu]=-[\mu]^T$, $[\mu]$ can be eliminated by adding Eq. (9) and its transpose

$$-4([\hat{I}]-[\gamma])+[k]\odot\left([\lambda][\phi]^T+[\phi][\lambda]^T\right)=[0] \tag{10}$$

Next, we pre-element-by-element multiply Eq. (10) by $\frac{1}{4}[k]$ and rearrange terms to obtain

$$[k]\odot[\gamma]=[k]-\tfrac{1}{4}[\Phi]\odot\left([\lambda][\phi]^T+[\phi][\lambda]^T\right) \tag{11}$$

where

$$[\Phi]=[k]\odot[k] \tag{12}$$

Substituting Eq. (11) into Eq. (5), we obtain

$$[A]+\{[\Phi]\odot([\lambda][\phi]^T)\}[\phi]$$
$$+\{[\Phi]\odot([\phi][\lambda]^T)\}[\phi]=[0] \tag{13}$$

where

$$[A]=4([M][\phi][\omega_n^2]-[k][\phi]) \tag{14}$$

Equation (13) can now be used to establish $[\lambda]$ which, when substituted into Eq. (11), yields the desired stiffness matrix [see Eq. (1)].

By performing the operations indicated in Eq. (13) and taking advantage of relationship 5 presented in the Appendix, it can be shown that

$$\{\bar{A}\}=([\alpha]+[\beta])\{\bar{\lambda}\} \tag{15}$$

where the elements of $[A]$ and $[\lambda]$ have been written as column vectors $\{\bar{A}\}$ and $\{\bar{\lambda}\}$, respectively. For example

$$[A]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32}\end{bmatrix}\rightarrow\{\bar{A}\}=\begin{Bmatrix} A_{11} \\ A_{12} \\ A_{21} \\ A_{22} \\ A_{31} \\ A_{32}\end{Bmatrix} \tag{16}$$

and

$$[\lambda]=\begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \\ \lambda_{31} & \lambda_{32}\end{bmatrix}\rightarrow\{\bar{\lambda}\}=\begin{Bmatrix} \lambda_{11} \\ \lambda_{12} \\ \lambda_{21} \\ \lambda_{22} \\ \lambda_{31} \\ \lambda_{32}\end{Bmatrix} \tag{17}$$

With $\{\bar{A}\}$ and $\{\bar{\lambda}\}$ defined as in Eqs. (16) and (17), it can be shown that $[\alpha]$ and $[\beta]$ are as follows:

$$[\alpha]=\left[\begin{array}{c|c|c|c|c|c} [G^1] & 0 & \cdots & 0 & \cdots & 0 \\ \hline 0 & [G^2] & & 0 & \cdots & 0 \\ \hline \vdots & & \ddots & & & \vdots \\ \hline 0 & 0 & & [G^i] & & 0 \\ \hline \vdots & \vdots & & & \ddots & \\ \hline 0 & 0 & \cdots & 0 & & [G^n] \end{array}\right] \tag{18}$$

where $[G^i]=-[\phi]^T[\hat{\Phi}^i][\phi]$ and $[\hat{\Phi}^i]$ is a diagonal matrix whose diagonal terms are the ith row of $[\Phi]$ [see Eq. (12)] and

$$[\beta]=\left[\begin{array}{c|c|c|c|c|c} [H]_{11} & [H]_{12} & \cdots & [H]_{1j} & \cdots & [H]_{1n} \\ \hline [H]_{21} & [H]_{22} & \cdots & [H]_{2j} & \cdots & [H]_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots & & \vdots \\ \hline [H]_{i1} & [H]_{i2} & & [H]_{ij} & \cdots & [H]_{in} \\ \hline \vdots & \vdots & & \vdots & \ddots & \vdots \\ \hline [H]_{n1} & [H]_{n2} & & [H]_{nj} & & [H]_{nn} \end{array}\right] \tag{19}$$

where

$$[H]_{ij}=-\{\bar{\phi}\}_j\{D^j\}_i^T$$
$$\{\bar{\phi}\}_j=j\text{th column of }[\phi]^T$$
$$\{D^j\}_i=i\text{th column of }[\phi]^T[\hat{\Phi}^j]$$

Equation (15) can now be used to establish $\{\bar{\lambda}\}$. Note that the rank of $[\alpha]+[\beta]$ will not exceed a number equal to the number of diagonal terms in $[k]$ plus all the nonzero coefficients on one side of the diagonal. The exact rank of $[\alpha]+[\beta]$ will depend on the structural connectivity and the number of independent constraints defined by Eq. (5). For most problems of practical interest, $[\alpha]+[\beta]$ will be singular. In addition, $[\alpha]$ and $[\beta]$ are symmetric and $[\alpha]+[\beta]$ is indefinite, i.e., the eigenvalues of $[\alpha]+[\beta]$ can be negative, zero, and/or positive.

Equation (15) can be solved by first defining the following transformation:

$$\{\bar{\lambda}\}=[\psi]\{\rho\} \tag{20}$$

where the columns of $[\psi]$ are the eigenvectors associated with the nonzero eigenvalues of $[\alpha]+[\beta]$. Substituting Eq. (20) into Eq. (15) and premultiplying by $[\psi]^T$, we obtain

$$[\psi]^T\{\bar{A}\}=[\psi]^T([\alpha]+[\beta])[\psi]\{\rho\}$$
$$=[\Omega]\{\rho\} \tag{21}$$

where $[\Omega]$ is a diagonal matrix whose diagonal terms are the nonzero eigenvalues of $[\alpha]+[\beta]$. Using Eq. (21) to solve for $\{\rho\}$ and substituting into Eq. (20) we finally obtain

$$\{\bar{\lambda}\}=[\psi][\Omega]^{-1}[\psi]^T\{\bar{A}\} \tag{22}$$

We can now construct $[\lambda]$ [see Eq. (17)] and establish $[K]$ using Eqs. (11) and (1).

The transformation defined by Eq. (20) yields a solution [Eq. (22)] that is applicable when the number of constraints defined by Eq. (5) exceeds the number of independent stiffness coefficients available for adjustment. If normal modes are used, the constraints will be consistent with each other and the true properties of the structure. Thus, the inclusion of additional modes, past a certain threshold, will not alter the identified stiffness coefficients.

This will not be the case if imperfect test modes are used. Because of measurement error, empirical modes will not necessarily be perfectly consistent with each other and the true properties of the structure. However, Eq. (22) will still provide a solution. Discussion of this feature is beyond the intended scope of this presentation. For the purposes of the present discussion, if we are dealing with test modes, we shall restrict our attention to the classical constrained minimization problem in which the number of constraints does not exceed the number of coefficients available for adjustment.

Demonstration of Procedure

The KMA procedure will be demonstrated by numerical simulation of a test problem. The procedure will be used to adjust the corrupted stiffness matrix of an eight degree of freedom (dof) analytical structure. The adjustments will first be performed using the normal modes of the system. The problem will then be repeated using simulated test modes.

A schematic representation of the structure is shown in Fig. 1, where the squares represent degrees of freedom and the springs represent load paths. The load path stiffness values and the diagonal terms of the diagonal mass matrix are also shown in the figure. The nonzero, upper triangular, stiffness matrix coefficients of the structure are presented in Table 1

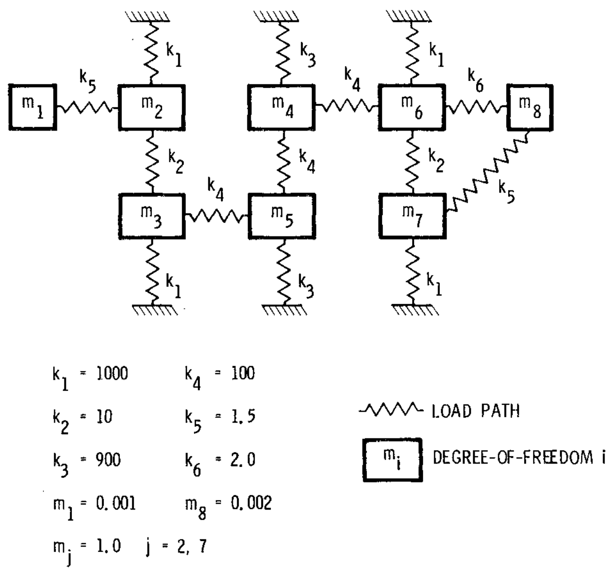

Fig. 1 Analytical test structure.

Table 1 Analytical test structure, nonzero, upper triangular stiffness matrix coefficients (normal modes)

		Adjusted stiffness coefficients			
Coefficient location	Corrupted stiffness coefficients	One normal mode	Two normal modes	Three normal modes	Exact stiffness coefficients
1,1	2.0	1.7	1.5	1.5	1.5
1,2	−2.0	−2.1	−1.5	−1.5	−1.5
2,2	1512.0	1013.5	1011.5	1011.5	1011.5
2,3	−10.0	−10.1	−10.0	−10.0	−10.0
3,3	1710.0	1275.8	1110.0	1110.0	1110.0
3,5	−200.0	−198.6	−100.0	−100.0	−100.0
4,4	850.0	1237.4	1100.0	1100.0	1100.0
4,5	−200.0	−178.6	−100.0	−100.0	−100.0
4,6	−200.0	−198.5	−100.0	−100.0	−100.0
5,5	850.0	1237.5	1100.0	1100.0	1100.0
6,6	1714.0	1279.7	1113.2	1112.0	1112.0
6,7	−10.0	−10.1	−9.0	−10.0	−10.0
6,8	−4.0	−4.1	−3.0	−2.0	−2.0
7,7	1512.0	1016.1	1012.4	1011.5	1011.5
7,8	−2.0	−2.0	−2.3	−1.5	−1.5
8,8	6.0	5.1	4.3	3.5	3.5

under the heading "exact stiffness coefficients." This structure represents a severe test case because of the large relative differences in the magnitudes of some of the stiffness matrix coefficients. In addition, all eight natural frequencies are within 27% of each other, and the first four natural frequencies are within 5% of each other.

The nonzero, upper triangular coefficients of the stiffness matrix that represent the best "analytical" model of the structure are presented in Table 1 under the heading of "corrupted stiffness coefficients." These coefficients were obtained by modifying the exact stiffness coefficients. Some values were increased by a maximum of 100%, others were not changed, and two diagonal terms were decreased by nearly 30%. By comparing the two sets of coefficients, it can be ascertained that, except for connectivity, the "analytical" stiffness matrix does not resemble the true stiffness matrix.

Before we proceed with the matrix adjustment, the number of independent coefficients in the "analytical" stiffness matrix will be established. As discussed previously, this can be accomplished by adding the number of diagonal terms in the matrix and the number of nonzero coefficients on one side of the diagonal. For the test problem, 16 independent terms are obtained.

Initially, it appears that all 16 stiffness coefficients can be identified from two normal modes. However, as illustrated in Fig. 1, dof 6, 7, and 8 involve a load path indeterminacy of three. Thus, to obtain exact identification, three normal modes will be needed. If less than three normal modes are used, an adjusted stiffness matrix will be obtained that minimizes the error function [Eq. (3)] and satisfies the symmetry [Eq. (6)], eigenproblem [Eq. (5)], and connectivity [Eq. (1)] constraints. To demonstrate this, the "analytical" stiffness matrix was first adjusted using only one normal mode, then two normal modes, and finally three normal modes.

The adjusted, nonzero, upper triangular stiffness matrix coefficients obtained using one normal mode are presented in Table 1. Comparing these coefficients to the exact stiffness coefficients, also shown in the table, we observe that many of the "analytical" stiffness matrix coefficients have significantly approached the true values. Also, connectivity has been preserved.

The adjusted, nonzero, upper triangular stiffness matrix coefficients obtained using the first two normal modes of the structure are also presented in Table 1. Again, connectivity has been preserved. In addition, the load path stiffnesses associated with dof 1-5 are in agreement with the exact stiffness coefficients. Furthermore, the terms associated with dof 6 and 7 have also improved substantially.

This two-mode case illustrates an interesting feature of the KMA procedure. The procedure will identify exactly those stiffness coefficients associated with the part of the structure for which the load path indeterminacy does not exceed the number of modes used. The stiffness coefficients associated with the part of the structure whose redundancy exceeds the number of modes used will be adjusted to minimize the error function of Eq. (3) and satisfy the connectivity, eigenproblem, and symmetry constraints [Eqs. (1), (5), and (6)].

As discussed previously, three normal modes will provide the needed conditions to exactly identify the stiffness matrix coefficients. Note that regardless of whether three, four, or all eight modes are used, $[\alpha]+[\beta]$ will yield only 16 nonzero eigenvalues, and the resulting stiffness matrices will all be identical.

The nonzero, upper triangular stiffness matrix coefficients identified with three normal modes are also presented in Table 1. By comparing these coefficients to the exact values, we observe that they are identical. Thus, with a subset of the system normal modes and knowledge of the structure's connectivity, the exact stiffness matrix coefficients have been identified.

The natural frequencies associated with each of the adjusted stiffness matrices are compared in Table 2 to the exact values. The correspondence between mode shapes and associated frequencies was established by performing a cross-orthogonality check between the true normal modes and the modes associated with each of the adjusted stiffness matrices. As can be ascertained from the table, the natural frequencies of the modes used in the identification are reproduced exactly by the adjusted model. A comparison of corresponding mode shapes also showed exact agreement. Furthermore, for the one- and two-mode cases, the modes and frequencies that were not used in the identification also are in closer agreement with the true values. The three-mode case, as expected, provides exact agreement in all modes.

To be of practical use, the KMA procedure must yield reasonable results when imperfect test-derived modes are used. To study the procedure's sensitivity to imperfect data, the applicable test cases were repeated using simulated test modes. The test modes were obtained by establishing the closed form quadrature response of the structure to multiple sinusoidally varying forces (a critical damping ratio of 0.01 was assigned each mode). The multiple force levels were adjusted to obtain "measured" modes whose mutual contamination did not ex-

Table 2 Comparison of natural frequencies, rad/s

Corrupted stiffness matrix	Adjusted stiffness matrix: One mode	Two modes	Three modes	Exact stiffness matrix
24.769	30.665[a]	30.665[a]	30.665[a]	30.665
38.776	31.737	31.711[a]	31.711[a]	31.711
38.859	31.850	31.763	31.763[a]	31.763
31.529	33.694	32.362	32.362	32.362
41.783	37.317	34.194	34.193	34.193
42.015	39.443	35.561	35.560	35.560
44.811	41.393	38.789	38.789	38.789
54.841	50.590	46.412	41.888	41.888

[a]Modes used to adjust corrupted stiffness matrix.

Table 3 Generalized mass matrix, simulated test modes

Test mode	Test mode 1	2	3	4	5
1	1.00	0.07	0.05	0.07	0.02
2		1.00	0.06	0.07	0.04
3			1.00	0.05	0.05
4				1.00	0.08
5					1.00

ceed approximately 10%. The unit-normalized generalized mass matrix for the first five modes is presented in Table 3. Experience with numerous complex spacecraft structures indicates that this level of contamination is what can be generally expected from a properly performed mode survey test.

It had been indicated earlier that if the mode shapes exhibit the orthogonality associated with normal modes [Eq. (7)] then the adjusted model will exactly reproduce the modes used in the identification. This was the case in the above example. However, because of limitations in experimental apparatus and inaccuracies in the analytical mass matrix, Eq. (7), in practice, will not be satisfied exactly. The analyst, therefore, has two choices. The adjustments can be performed with mode shapes and a mass matrix that do not satisfy Eq. (7). The adjusted stiffness matrix will nearly satisfy Eq. (5), however, the new model will not necessarily reproduce the modes used in the identification. The second choice is to analytically adjust either the empirical modes, the mass matrix, or both such that Eq. (7) is satisfied, and then perform the identification.

Presently, for many complex spacecraft structures, it is possible to formulate analytical mass matrices and measure mode shapes such that accurate analytical model adjustments may be possible. It has become a widely accepted practice to judge the quality of measured modes, and to some extent the accuracy of the mass matrix, by calculating the unit-normalized generalized mass matrix and comparing the magnitudes of the off-diagonal terms to a predetermined value. It is generally accepted that modes of acceptable quality have been measured if these off-diagonal terms are less than 0.10. It should be noted, however, that additional research is needed to determine whether this criterion yields modes of sufficient accuracy for identification purposes.

To obtain mode shapes that satisfy Eq. (7), only small changes are generally required in the measured mode vectors if they satisfy the 0.10 orthogonality criterion.[17] Therefore, for the purposes of the present discussion we will analytically orthogonalize the "test" modes before performing the stiffness matrix adjustments. To orthogonalize the "test" modes, the symmetric correction matrix orthogonalization procedure described in Ref. 17 will be used, i.e.,

$$[\phi^c]=[\phi^m]\left([\phi^m]^T[M][\phi^m]\right)^{-\frac{1}{2}} \tag{23}$$

Table 4 Comparison of mode shapes

	"Test" mode shapes	Orthogonalized "test" mode shapes	Normal mode shapes
$\{\phi\}_1$	0.029	0.112	0.144
	0.015	0.042	0.054
	0.374	0.356	0.360
	0.595	0.609	0.606
	0.604	0.600	0.605
	0.356	0.365	0.362
	0.117	0.076	0.062
	0.551	0.522	0.505
$\{\phi\}_2$	2.753	2.774	2.776
	0.909	0.917	0.915
	0.098	0.138	0.127
	−0.080	−0.078	−0.088
	0.027	0.050	0.041
	−0.127	−0.123	−0.124
	−0.365	−0.330	−0.337
	−0.539	−0.498	−0.506

Table 5 Analytical test structure, nonzero, upper triangular stiffness matrix coefficients ("test" modes)

Coefficient location	Corrupted stiffness coefficients	Adjusted stiffness coefficients: One "test" mode	Two "test" modes	Exact stiffness coefficients
1,1	2.0	1.7	1.5	1.5
1,2	−2.0	−2.1	−1.4	−1.5
2,2	1512.0	1030.4	1010.9	1011.5
2,3	−10.0	−10.1	−8.0	−10.0
3,3	1710.0	1276.6	1091.0	1110.0
3,5	−200.0	−198.6	−88.8	−100.0
4,4	850.0	1235.2	1098.1	1100.0
4,5	−200.0	−178.6	−99.6	−100.0
4,6	−200.0	−198.5	−99.6	−100.0
5,5	850.0	1239.3	1094.0	1100.0
6,6	1714.0	1279.8	1113.5	1112.0
6,7	−10.0	−10.0	−11.9	−10.0
6,8	−4.0	−4.1	−3.1	−2.0
7,7	1512.0	1002.1	1013.6	1011.5
7,8	−2.0	−2.0	−2.4	−1.5
8,8	6.0	5.1	4.4	3.5

where $[\phi^m]$ are the measured modes and $[\phi^c]$ the orthogonalized modes. Only the first five modes were included in the orthogonalization, whereas all eight modes contributed in the response calculations. This simulates actual test conditions where the modes not measured contaminate the measured modes. The two orthogonalized modes used in the subsequent stiffness matrix adjustments are compared in Table 4 to the "test" modes and the true normal modes.

The adjusted, nonzero, upper triangular stiffness matrix coefficients obtained using the first orthogonalized "test" mode are presented in Table 5. Comparing these coefficients to the exact values, we observe that many of the "analytical" stiffness matrix coefficients have improved substantially, particularly the diagonal terms. When the adjusted matrix coefficients are compared to those obtained with a single normal mode (Table 1), it can be observed that they are comparable. In addition, connectivity has been preserved.

The adjusted, nonzero, upper triangular stiffness matrix coefficients obtained with two orthogonalized "test" modes are also presented in Table 5. By comparing the adjusted coefficients to those in Table 1, we observe comparable improvement to that obtained with two normal modes. Also, as has been the case for all test problems, connectivity has been preserved. Thus, with a limited number of mode shapes, knowledge of the structural connectivity, and a physically realistic minimization function, it is possible to improve a deficient stiffness matrix.

Some Practical Considerations

The minimization error function [Eq. (3)] used in the KMA procedure minimizes the percentage change to each stiffness coefficient. Thus, dramatic changes will occur only if dictated by the eigenproblem [Eq. (5)] and connectivity constraints [Eq. (1)]. Experience to date indicates that the procedure can yield unrealistic stiffness coefficients changes if the mode vectors contain relatively large errors, particularly sign errors. Therefore, it is suggested that before an adjusted stiffness matrix is accepted as valid, all adjustments should be reconciled by a review of drawings and modeling methodology.

Generally, analytical dynamic model stiffness matrices are obtained by Gaussian elimination reduction of detailed structural models. This reduction procedure can yield fully populated stiffness matrices. Inspection of these matrices generally reveals that a large percentage of the coefficients are relatively small numbers. In many structures, it is possible to remove these "weak" load paths without perturbing the dynamic properties of interest. By increasing the number of coefficients with values of zero, the number of stiffness coefficients available for adjustment is reduced. It is expected that in many instances this will enhance the performance of the KMA procedure, particularly if a limited number of modes is available.

An additional practical consideration is the cost of extracting the eigendata from $[\alpha]+[\beta]$. The size of $[\alpha]+[\beta]$ will be degrees-of-freedom times the number of modes used in the identification, which for many structures will be prohibitively large (present computer capabilities limit the size of $[\alpha]+[\beta]$ to an order of approximately one to two thousand). Therefore, if post-test adjustments are contemplated, the size of the dynamic model and, indeed, the test article configuration (e.g., one structure test vs two substructure tests) should all be part of the overall dynamic model formulation plan.

Summary

A procedure has been introduced that uses, in addition to mode data, structural connectivity information to optimally adjust deficient stiffness matrices. The procedure was developed using constrained minimization theory. The minimization error function was formulated such that the resulting changes to stiffness coefficients are a minimum. The resulting procedure retains the physical configuration of the analytical model, and the adjusted model exactly reproduces the modes used in the identification.

The stiffness matrix adjustment (KMA) procedure was demonstrated by numerical simulation of a test problem. The procedure was used to adjust the corrupted stiffness matrix of an eight degree of freedom analytical structure. The adjustments were first performed using the system normal modes, of which only three were needed for exact identification. The test problem was then repeated, using simulated test modes, with excellent results.

Appendix

The derivation of the KMA procedure used an element-by-element matrix multiplication operator $\odot$. Relevant properties of the operator are presented below.

1) $$[c]=[a]\odot[b]$$

defines $c_{ij}=a_{ij}b_{ij}$

2) $$[c]=[d]([a]\odot[b])$$

implies $c_{ij}=\sum_{k=1}^{n} d_{ik}(a_{kj}b_{kj})$

3) $$[a]\odot[b]=[b]\odot[a]$$

4) $$\{[d]([a]\odot[b])\}^T=([a]^T\odot[b]^T)[d]^T$$

5) $$\underset{n\times n}{[a]}\odot\underset{n\times n}{[b]}=\sum_{j=1}^{n}[\hat{a}]_j[\hat{b}]_j$$

and

$$\hat{a}_{lk}=a_{jk} \quad \text{for } l=j \qquad \hat{b}_{lk}=b_{jk} \quad \text{for } l=k$$
$$=0 \quad \text{for } l\neq j \qquad =0 \quad \text{for } l\neq k$$

Example:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\odot\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix} b_{11} & 0 \\ 0 & b_{12}\end{bmatrix}+\begin{bmatrix} 0 & 0 \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix} b_{21} & 0 \\ 0 & b_{22}\end{bmatrix}$$

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